UNIVERSITY OF WEST BOHEMIA

Applied and Computational Mechanics 2 (2008) 113–122

Applied and Computational Mechanics

Contribution to stress sensitivity analysis of the shell finite elements

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Received 28 August 2008; received in revised form 6 October 2008

Abstract

The sensitivity analysis and the finite elements method represent an important tool for the influence analysis of the structural parameters. This analysis plays a significant role in the decision process of the formulation of the structural optimizing or probability analysis. The goal of the paper is to present theoretic and numerical aspects of the shell element stress sensitivity analysis with the respect to the thickness and its implementation into finite element code MATFEM inbuilt to Matlab.

Keywords: shell element, stress sensitivity analysis, structural optimization

1. Introduction

Nowadays the sensitivity analysis is a significant tool helping to realize a structural parameters influence analysis. This analysis is usually very computer time consuming but the results are very innovative. This process is often applied to a structural analysis, i.e. in stress and strain analysis, modal and spectral or buckling analysis, stochastic analysis and so on [3, 4].

Application of the sensitivity analysis is not associated only with the structural optimizing but also with the analysis of the mechanical systems with uncertain parameters, mainly in the usage of so-called perturbation methods based on differentiation of the response with respect to the uncertain system parameters (stiffness, mass, damping, etc.). Implementation of this computational process into the finite element method has characterised mainly the era of development of structural optimising techniques in eighties.

The finite element modelling of box, shell or thin-walled structures are usually realised using thin shell finite elements (Kirchhoff's or Mindlin's formulation) [1, 2, 8, 9]. The stiffness parameters depend on material constants and element geometry, mainly on its thickness. Therefore, the thickness will be the variable in the following theoretical and numerical stress sensitivity analysis of the shell finite element; the fundamental information about this analysis can be found in Appendix.

2. Element stress analysis

The stress calculation is based on the expression of the element membrane forces and bending moments (without the shear forces) [2, 5], i.e.

$$[F_{xx} \quad F_{yy} \quad F_{xy}]^T = \mathbf{F}_m = \int_S \mathbf{E}_m \cdot \boldsymbol{\varepsilon}_m \, \mathrm{d}S = \mathbf{E}_m \cdot \int_S \mathbf{B}_m \, \mathrm{d}S \cdot \mathbf{u}_{el} = t \cdot \mathbf{D} \cdot \mathbf{I}_m \cdot \mathbf{u}_{el} \tag{1}$$

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and

$$[M_{xx} \quad M_{yy} \quad M_{xy}]^T = \mathbf{M}_b = \int_S \mathbf{E}_b \cdot \boldsymbol{\varepsilon}_b \, \mathrm{d}S = \mathbf{E}_b \cdot \int_S \mathbf{B}_b \, \mathrm{d}S \cdot \mathbf{u}_{el} = \frac{t^3}{12} \cdot \mathbf{D} \cdot \mathbf{I}_b \cdot \mathbf{u}_{el}\,. \tag{2}$$

The integration matrices $\mathbf{I}_m$ and $\mathbf{I}_b$ are

$$\mathbf{I}_m = \int_S \mathbf{B}_m \, \mathrm{d}S, \qquad \mathbf{I}_b = \int_S \mathbf{B}_b \, \mathrm{d}S \tag{3}$$

and can be calculated only using the numerical approach. Further details about $\mathbf{E}_m$, $\mathbf{E}_b$, $\mathbf{D}$, $\mathbf{B}_m$, $\mathbf{B}_b$, $\mathbf{u}_{el}$ and t are presented in Appendix. The extreme stress values can be expected at the top or at the bottom surface. Generally, it means

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\sigma}_{mb}|_{top} \\ \boldsymbol{\sigma}_{mb}|_{bot} \end{bmatrix} &= \begin{Bmatrix} \sigma_{xx,top} \\ \sigma_{yy,top} \\ \sigma_{xy,top} \\ \sigma_{xx,bot} \\ \sigma_{yy,bot} \\ \sigma_{xy,bot} \end{Bmatrix} = \begin{bmatrix} \frac{1}{t} & 0 & 0 & \frac{6}{t^2} & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 & \frac{6}{t^2} & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 & \frac{6}{t^2} \\ \frac{1}{t} & 0 & 0 & \frac{-6}{t^2} & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 & \frac{-6}{t^2} & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 & \frac{-6}{t^2} \end{bmatrix} \cdot \begin{Bmatrix} F_{xx} \\ F_{yy} \\ F_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \\ &= \begin{bmatrix} \mathbf{A}_{t,top} \\ \mathbf{A}_{t,bot} \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{F}_m \\ \mathbf{M}_b \end{Bmatrix}. \end{aligned} \tag{4}$$

Stresses at the top surface may be expressed as

$$\boldsymbol{\sigma}_{mb}|_{top} = \mathbf{A}_{t,top} \cdot \begin{Bmatrix} \mathbf{F}_m \\ \mathbf{M}_b \end{Bmatrix} \tag{5}$$

with

$$\mathbf{A}_{t,top} = \begin{bmatrix} \frac{1}{t} & 0 & 0 & \frac{6}{t^2} & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 & \frac{6}{t^2} & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 & \frac{6}{t^2} \end{bmatrix} \tag{6}$$

and at the bottom surface

$$\boldsymbol{\sigma}_{mb}|_{bot} = \mathbf{A}_{t,bot} \cdot \begin{Bmatrix} \mathbf{F}_m \\ \mathbf{M}_b \end{Bmatrix} \tag{7}$$

with

$$\mathbf{A}_{t,bot} = \begin{bmatrix} \frac{1}{t} & 0 & 0 & \frac{-6}{t^2} & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 & \frac{-6}{t^2} & 0 \\ 0 & 0 & \frac{1}{t} & 0 & 0 & \frac{-6}{t^2} \end{bmatrix}. \tag{8}$$

Let's build new material and integral matrices

$$\mathbf{E}_{mb} = \begin{bmatrix} t \cdot \mathbf{I}_3 & \mathbf{0}_3 \\ \mathbf{0}_3 & \frac{t^3}{12} \cdot \mathbf{I}_3 \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{D} \\ \mathbf{D} \end{Bmatrix} = \mathbf{D}_t \cdot \mathbf{D}_{mb}, \qquad \mathbf{I}_{mb} = \begin{Bmatrix} \mathbf{I}_m \\ \mathbf{I}_b \end{Bmatrix}, \tag{9}$$

where matrix $\mathbf{I}_3$ is the unit matrix. Then (5) and (7) can be written as follows

$$\boldsymbol{\sigma}_{mb}|_{top} = \mathbf{A}_{t,top} \cdot \mathbf{E}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el} = \mathbf{A}_{t,top} \cdot \mathbf{D}_t \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el}, \tag{10}$$

$$\boldsymbol{\sigma}_{mb}|_{bot} = \mathbf{A}_{t,bot} \cdot \mathbf{E}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el} = \mathbf{A}_{t,bot} \cdot \mathbf{D}_t \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el}. \tag{11}$$

Generally, the top or bottom von Mises stresses may be calculated from relations

$$\sigma^2_{ekv}|_{top} = \boldsymbol{\sigma}^T_{mb}|_{top} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{mb}|_{top} \qquad \text{or} \qquad \sigma^2_{ekv}|_{bot} = \boldsymbol{\sigma}^T_{mb}|_{bot} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{mb}|_{bot} \tag{12}$$

where

$$\mathbf{T}_{mb} = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \tag{13}$$

Using (10) and (11) in (12) we obtain

$$\begin{aligned} \sigma^2_{ekv}|_{top} &= \boldsymbol{\sigma}^T_{mb}|_{top} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{mb}|_{top} = \\ &= \mathbf{u}^T_{el} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{D}^T_{t} \cdot \mathbf{A}^T_{t,top} \cdot \mathbf{T}_{mb} \cdot \mathbf{A}_{t,top} \cdot \mathbf{D}_{t} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el} = \\ &= \mathbf{u}^T_{el} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{T}_{t,top} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \sigma^2_{ekv}|_{bot} &= \boldsymbol{\sigma}^T_{mb}|_{bot} \cdot \mathbf{T}_{mb} \cdot \boldsymbol{\sigma}_{mb}|_{bot} = \\ &= \mathbf{u}^T_{el} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{D}^T_{t} \cdot \mathbf{A}^T_{t,bot} \cdot \mathbf{T}_{mb} \cdot \mathbf{A}_{t,bot} \cdot \mathbf{D}_{t} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el} = \\ &= \mathbf{u}^T_{el} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{T}_{t,bot} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{u}_{el} \end{aligned} \tag{15}$$

where

$$\mathbf{T}_{t,top} = \begin{bmatrix} 1 & -0.5 & 0 & 0.5\cdot t & -0.25\cdot t & 0 \\ -0.5 & 1 & 0 & -0.25\cdot t & 0.5\cdot t & 0 \\ 0 & 0 & 3 & 0 & 0 & 1.5\cdot t \\ 0.5\cdot t & -0.25\cdot t & 0 & 0.25\cdot t^2 & -0.125\cdot t^2 & 0 \\ -0.25\cdot t & 0.5\cdot t & 0 & -0.125\cdot t^2 & 0.25\cdot t^2 & 0 \\ 0 & 0 & 1.5\cdot t & 0 & 0 & 0.75\cdot t^2 \end{bmatrix} \tag{16}$$

and

$$\mathbf{T}_{t,bot} = \begin{bmatrix} 1 & -0.5 & 0 & -0.5\cdot t & 0.25\cdot t & 0 \\ -0.5 & 1 & 0 & 0.25\cdot t & -0.5\cdot t & 0 \\ 0 & 0 & 3 & 0 & 0 & -1.5\cdot t \\ -0.5\cdot t & 0.25\cdot t & 0 & 0.25\cdot t^2 & -0.125\cdot t^2 & 0 \\ 0.25\cdot t & -0.5\cdot t & 0 & -0.125\cdot t^2 & 0.25\cdot t^2 & 0 \\ 0 & 0 & -1.5\cdot t & 0 & 0 & 0.75\cdot t^2 \end{bmatrix}. \tag{17}$$

Assuming a relation between the local element displacements $\mathbf{u}_{el}$ and the global displacement vector $\mathbf{u}$

$$\mathbf{u}_{el} = \mathbf{T}_{LG} \cdot \mathbf{T}_{01} \cdot \mathbf{u}, \tag{18}$$

(14) and (15) may be rewritten as

$$\sigma^2_{ekv}|_{top} = \mathbf{u}^T \cdot \mathbf{T}^T_{01} \cdot \mathbf{T}^T_{LG} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{T}_{t,top} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{T}_{LG}\mathbf{T}_{01}\mathbf{u} \tag{19}$$

and

$$\sigma^2_{ekv}|_{bot} = \mathbf{u}^T \cdot \mathbf{T}^T_{01} \cdot \mathbf{T}^T_{LG} \cdot \mathbf{I}^T_{mb} \cdot \mathbf{D}^T_{mb} \cdot \mathbf{T}_{t,bot} \cdot \mathbf{D}_{mb} \cdot \mathbf{I}_{mb} \cdot \mathbf{T}_{LG}\mathbf{T}_{01}\mathbf{u} \tag{20}$$

where $\mathbf{T}_{LG}$ is a "classic" transformation matrix between the local and the global coordinate systems, $\mathbf{T}_{01}$ is a Boolean matrix, i.e. the localization matrix determining the element position in the global stiffness matrix.

3. Stress sensitivity analysis

The stress sensitivity analysis means finding of von Mises stress derivative with the respect to a chosen structural parameter, in our case the thickness t. Let's analyse the differentiation of von Mises stress of j*-th* element with respect to i*-th* element thickness t_i. Applying (19) we can obtain

– for $i = j$

$$\begin{aligned}\frac{\partial\sigma^2_{i\_ekv}|_{top}}{\partial t_i} = {} & \frac{\partial\mathbf{u}^T}{\partial t_i}\cdot\mathbf{T}^T_{i\_01}\cdot\mathbf{T}^T_{i\_LG}\cdot\mathbf{I}^T_{i\_mb}\cdot\mathbf{D}^T_{i\_mb}\cdot\mathbf{T}_{i\_t,top}\cdot\mathbf{D}_{i\_mb}\cdot\mathbf{I}_{i\_mb}\cdot\mathbf{T}_{i\_LG}\cdot\mathbf{T}_{i\_01}\cdot\mathbf{u}+\\ & +\mathbf{u}^T\cdot\mathbf{T}^T_{i\_01}\cdot\mathbf{T}^T_{i\_LG}\cdot\mathbf{I}^T_{i\_mb}\cdot\mathbf{D}^T_{i\_mb}\cdot\frac{\partial\mathbf{T}_{i\_t,top}}{\partial t_i}\cdot\mathbf{D}_{i\_mb}\cdot\mathbf{I}_{i\_mb}\cdot\mathbf{T}_{i\_LG}\cdot\mathbf{T}_{i\_01}\cdot\mathbf{u}+\\ & +\mathbf{u}^T\cdot\mathbf{T}^T_{i\_01}\cdot\mathbf{T}^T_{i\_LG}\cdot\mathbf{I}^T_{i\_mb}\cdot\mathbf{D}^T_{i\_mb}\cdot\mathbf{T}_{i\_t,top}\cdot\mathbf{D}_{i\_mb}\cdot\mathbf{I}_{i\_mb}\cdot\mathbf{T}_{i\_LG}\cdot\mathbf{T}_{i\_01}\cdot\frac{\partial\mathbf{u}}{\partial t_i}\end{aligned} \quad (21)$$

$$\begin{aligned}\frac{\partial\sigma^2_{i\_ekv}|_{bot}}{\partial t_i} = {} & \frac{\partial\mathbf{u}^T}{\partial t_i}\cdot\mathbf{T}^T_{i\_01}\cdot\mathbf{T}^T_{i\_LG}\cdot\mathbf{I}^T_{i\_mb}\cdot\mathbf{D}^T_{i\_mb}\cdot\mathbf{T}_{i\_t,bot}\cdot\mathbf{D}_{i\_mb}\cdot\mathbf{I}_{i\_mb}\cdot\mathbf{T}_{i\_LG}\cdot\mathbf{T}_{i\_01}\cdot\mathbf{u}+\\ & +\mathbf{u}^T\cdot\mathbf{T}^T_{i\_01}\cdot\mathbf{T}^T_{i\_LG}\cdot\mathbf{I}^T_{i\_mb}\cdot\mathbf{D}^T_{i\_mb}\cdot\frac{\partial\mathbf{T}_{i\_t,bot}}{\partial t_i}\cdot\mathbf{D}_{i\_mb}\cdot\mathbf{I}_{i\_mb}\cdot\mathbf{T}_{i\_LG}\cdot\mathbf{T}_{i\_01}\cdot\mathbf{u}+\\ & +\mathbf{u}^T\cdot\mathbf{T}^T_{i\_01}\cdot\mathbf{T}^T_{i\_LG}\cdot\mathbf{I}^T_{i\_mb}\cdot\mathbf{D}^T_{i\_mb}\cdot\mathbf{T}_{i\_t,bot}\cdot\mathbf{D}_{i\_mb}\cdot\mathbf{I}_{i\_mb}\cdot\mathbf{T}_{i\_LG}\cdot\mathbf{T}_{i\_01}\cdot\frac{\partial\mathbf{u}}{\partial t_i}\end{aligned} \quad (22)$$

– for $i \neq j$

$$\begin{aligned}\frac{\partial\sigma^2_{j\_ekv}|_{top}}{\partial t_i} = {} & \frac{\partial\mathbf{u}^T}{\partial t_i}\cdot\mathbf{T}^T_{j\_01}\cdot\mathbf{T}^T_{j\_LG}\cdot\mathbf{I}^T_{j\_mb}\cdot\mathbf{D}^T_{j\_mb}\cdot\mathbf{T}_{j\_t,top}\cdot\mathbf{D}_{j\_mb}\mathbf{I}_{j\_mb}\cdot\mathbf{T}_{vLG}\cdot\mathbf{T}_{j\_01}\cdot\mathbf{u}+ \quad (23)\\ & +\mathbf{u}^T\cdot\mathbf{T}^T_{j\_01}\cdot\mathbf{T}^T_{j\_LG}\cdot\mathbf{I}^T_{j\_mb}\cdot\mathbf{D}^T_{j\_mb}\cdot\mathbf{T}_{j\_t,top}\cdot\mathbf{D}_{j\_mb}\cdot\mathbf{I}_{j\_mb}\cdot\mathbf{T}_{j\_LG}\cdot\mathbf{T}_{j\_01}\cdot\frac{\partial\mathbf{u}}{\partial t_i}\end{aligned}$$

$$\begin{aligned}\frac{\partial\sigma^2_{j\_ekv}|_{bot}}{\partial t_i} = {} & \frac{\partial\mathbf{u}^T}{\partial t_i}\cdot\mathbf{T}^T_{j\_01}\cdot\mathbf{T}^T_{j\_LG}\cdot\mathbf{I}^T_{j\_mb}\cdot\mathbf{D}^T_{j\_mb}\cdot\mathbf{T}_{j\_t,bot}\cdot\mathbf{D}_{j\_mb}\cdot\mathbf{I}_{j\_mb}\cdot\mathbf{T}_{vLG}\cdot\mathbf{T}_{j\_01}\cdot\mathbf{u}+ \quad (24)\\ & +\mathbf{u}^T\cdot\mathbf{T}^T_{j\_01}\cdot\mathbf{T}^T_{j\_LG}\cdot\mathbf{I}^T_{j\_mb}\cdot\mathbf{D}^T_{j\_mb}\cdot\mathbf{T}_{j\_t,bot}\cdot\mathbf{D}_{j\_mb}\cdot\mathbf{I}_{j\_mb}\cdot\mathbf{T}_{j\_LG}\cdot\mathbf{T}_{j\_01}\cdot\frac{\partial\mathbf{u}}{\partial t_i}\end{aligned}$$

where

$$\frac{\partial\mathbf{T}_{i\_t,top}}{\partial t_i} = \begin{bmatrix} 0 & 0 & 0 & 0.5 & -0.25 & 0\\ 0 & 0 & 0 & -0.25 & 0.5 & 0\\ 0 & 0 & 0 & 0 & 0 & 1.5\\ 0.5 & -0.25 & 0 & 0.5\cdot t_i & -0.25\cdot t_i & 0\\ -0.25 & 0.5 & 0 & -0.25\cdot t_i & 0.5\cdot t_i & 0\\ 0 & 0 & 1.5 & 0 & 0 & 1.5\cdot t_i\end{bmatrix} \quad (25)$$

and

$$\frac{\partial\mathbf{T}_{i\_t,bot}}{\partial t_i} = \begin{bmatrix} 0 & 0 & 0 & -0.5 & 0.25 & 0\\ 0 & 0 & 0 & 0.25 & -0.5 & 0\\ 0 & 0 & 0 & 0 & 0 & -1.5\\ -0.5 & 0.25 & 0 & 0.5\cdot t_i & -0.25\cdot t_i & 0\\ 0.25 & -0.5 & 0 & -0.25\cdot t_i & 0.5\cdot t_i & 0\\ 0 & 0 & -1.5 & 0 & 0 & 1.5\cdot t_i\end{bmatrix}. \quad (26)$$

The derivative $\mathbf{u}$ with the respect to t_i may be expressed as

$$\frac{\partial\mathbf{u}}{\partial t_i} = \mathbf{K}^{-1}\cdot\left(\frac{\partial\mathbf{f}}{\partial t_i} - \frac{\partial\mathbf{K}}{\partial t_i}\cdot\mathbf{u}\right) \quad (27)$$

or in more detail

$$\frac{\partial \mathbf{u}}{\partial t_i} = \mathbf{K}^{-1} \cdot \left[\frac{\partial \mathbf{f}}{\partial t_i} - \left(\sum_{j=1}^{n} \mathbf{T}_{j\_01}^T \cdot \mathbf{T}_{j\_LG}^T \cdot \frac{\partial(\mathbf{K}_{j\_m} + \mathbf{K}_{j\_b} + \mathbf{K}_{j\_s})}{\partial t_i} \cdot \mathbf{T}_{j\_LG} \cdot \mathbf{T}_{j\_01} \right) \cdot \mathbf{u} \right]. \quad (28)$$

The relation $\frac{\partial \mathbf{f}}{\partial t_i}$ is often zero and the derivative of the all element components of the stiffness matrix can be realized as follows [3]

$$\frac{\partial(\mathbf{K}_{i\_m} + \mathbf{K}_{i\_b} + \mathbf{K}_{i\_s})}{\partial t_i} = \left\langle \begin{array}{ll} \dfrac{1}{t_i} \cdot (\mathbf{K}_{i\_m} + 3 \cdot \mathbf{K}_{i\_b} + \mathbf{K}_{i\_s}), & j = i \\ \mathbf{0}, & j \neq i \end{array} \right. \quad (29)$$

The particular membrane, bending and shear matrices are presented in Appendix, equations (A13), (A15). More details can be found in [1, 2].

Finally, the derivative of the von Mises stress (at the top and at the bottom surfaces) with the respect to the element thickness t_i is following

$$\frac{\partial \sigma_{j\_ekv}|_{top}}{\partial t_i} = \frac{1}{2\sigma_{j\_ekv}|_{top}} \cdot \frac{\partial \sigma_{j\_ekv}^2|_{top}}{\partial t_i} \quad \text{and} \quad \frac{\partial \sigma_{j\_ekv}|_{bot}}{\partial t_i} = \frac{1}{2\sigma_{j\_ekv}|_{bot}} \cdot \frac{\partial \sigma_{j\_ekv}^2|_{bot}}{\partial t_i}. \quad (30)$$

All presented approaches have been implemented into Matlab's FE software MATFEM developed by the authors.

4. Numerical examples

Example 1

Determine the element stress derivative (eqs. 21, 22) with respect to the thickness t_1 and t_2 of the shell structure on figure 1. Let's consider the following input parameters: elasticity modulus $E = 3 \cdot 10^6$ MPa, Poisson's ratio $\mu = 0.3$, thicknesses $t_1 = 3$ mm and $t_2 = 2$ mm and force $F_Z = 2\,500$ N concentrated into each node of the top curved surface.

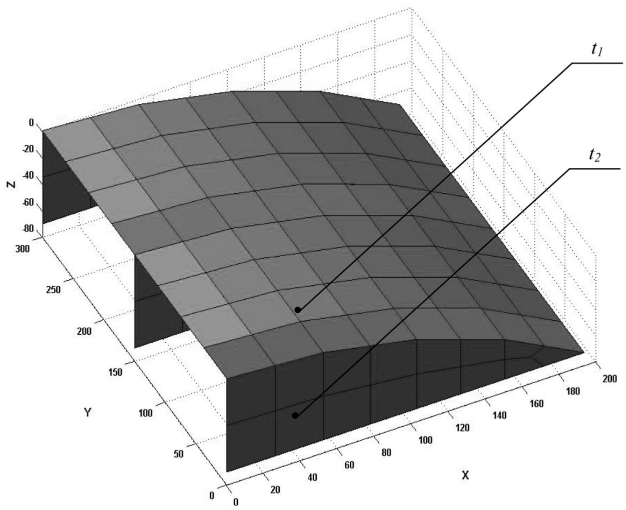

Fig. 1. Half model of the analysed shell structure in MATFEM

The chosen calculated values of the stress gradients are written in table 1. The presented analytic stress gradient calculation has been confronted with the "classic" numerical computational approach ($\Delta\sigma_j/\Delta t_i$). A graphic presentation of the stress gradients distribution in each of the elements is on figures 2 and 3.

Table 1. Stress gradient values for the chosen elements — analytical vs. numerical calculation

Nr. of element	Stress gradient with respect t_1		Nr. of element	Stress gradient with respect t_2	
	Analytically	Numerically		Analytically	Numerically
4	180.692 5	180.821 7	81	72.143 2	72.135 6
15	178.192 9	178.346 4	66	56.861 7	56.884 1
12	172.767 3	172.940 1	65	56.513 6	56.551 4
7	172.210 5	172.342 7	92	54.806 5	54.844 9
53	170.204 1	170.445 5	80	52.539 4	52.564 9

The results document the influence of both parameters on the stresses and the major signification of thickness t_1. This information may be used for the next optimizing process.

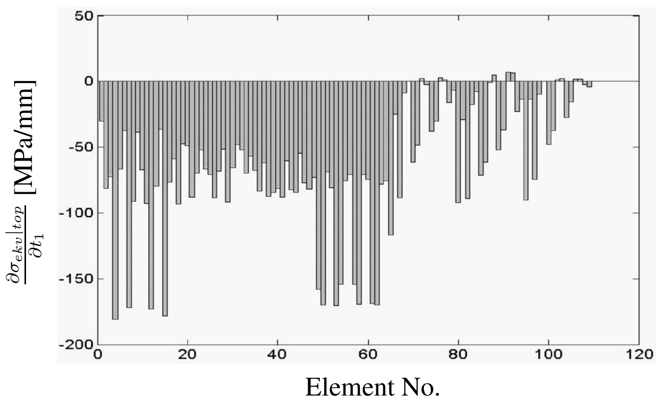

Fig. 2. Stress sensitivity with the respect to t_1

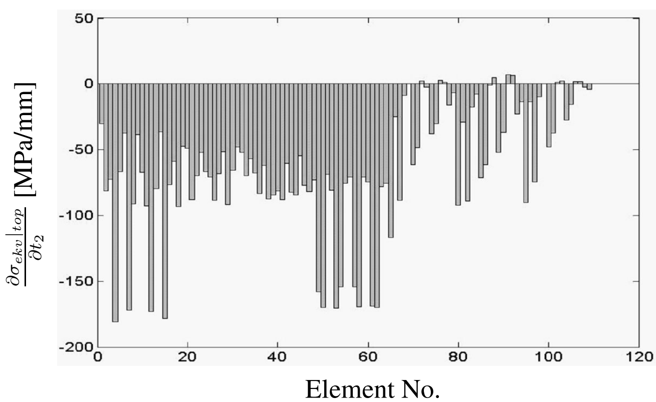

Fig. 3. Stress sensitivity with the respect to t_2

Example 2

Find out the optimal thickenesses t_1 and t_2 of the shell structure from the previous example. Let the searching process be based on the basis of the presented stress sensitivity analysis. Considering the stress limit $\sigma_{dov} = 200\,\text{MPa}$ it is possible to formulate the optimizing problem as follows

$$\text{Weight}(t_1, t_2) \rightarrow \min . \qquad \text{subject to} \qquad [\max \sigma_{ekv}(t_1, t_2)] - \sigma_{dov} \leq 0$$

The graphic presentation of this optimizing problem is on Fig. 4. Results are summarized in Tab. 2.

Table 2. Results of the optimizing process

t_1 [mm]	t_2 [mm]	Weight [kg]	Max. stress [MPa]
2.4	6.4	3.671 8	200,004

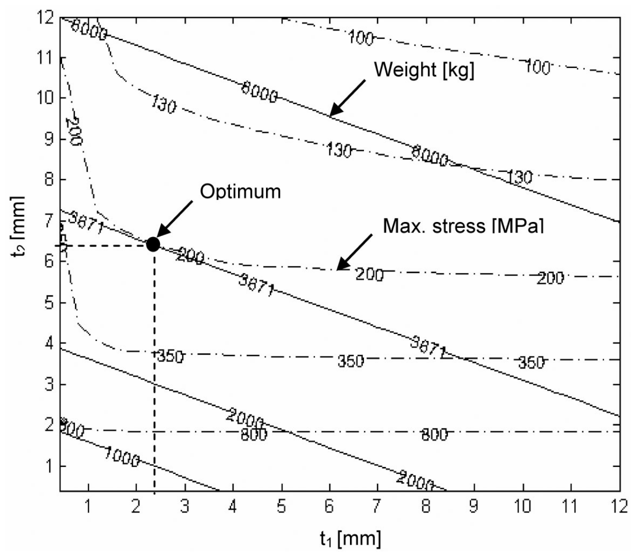

Fig. 4. Graphic presentation of the optimizing problem

5. Conclusion

The work presents an analytic approach to the stress sensitivity analysis of the shell finite element focused on its thickness. The whole computational procedure was inbuilt into Matlab's software module MATFEM. Testing examples support the authors' considerations about the effectiveness of the presented approach.

Acknowledgement

The work has been supported by the grant project VEGA 1/3168/06 and by the research project AV 4/2044/08.

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Appendix

Let's remember the well-known basic data about stiffness parameters calculation of the applied four-nodes thin shell finite element. This element belongs to a group of traditional finite elements therefore more details inhere in the relevant literature [1, 2, 5, 6, 7].

Generally, the virtual modelling of thin shell structures in the mechanical or civil engineering is based on the element whose isoparametric formulation has several advantages (e.g. a degeneration of the number of nodes from 4 to 3, appropriate for the automesh). The nodes are located on the midsurface and each node has 6 degrees of freedom (3 displacements and 3 rotational DOFs with a zero rotation about z-axis normal to the plane, see Fig. 5). The element contains a membrane, bending and shear stiffness parameters. The constant element thickness is considered.

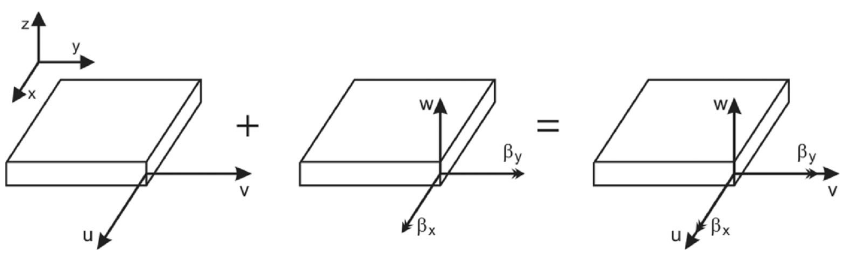

Fig. 5. Presentation of the displacement and rotational degrees of freedom

According to the Mindlin's theory, the displacement functions may be written in the form

$$u(x,y) = z \cdot \beta_x(x,y), \qquad v(x,y) = -z \cdot \beta_y(x,y), \qquad w = w(x,y). \tag{A1}$$

Using the well-known isoparametric approximation, we obtain

$$\begin{gathered} u = \sum_{i=1}^{4} N_i \cdot u_i, \qquad v = \sum_{i=1}^{4} N_i \cdot v_i, \qquad w = \sum_{i=1}^{4} N_i \cdot w_i, \\ \beta_x = \sum_{i=1}^{4} N_i \cdot \beta_{xi}, \qquad \beta_y = \sum_{i=1}^{4} N_i \cdot \beta_{yi}, \end{gathered} \tag{A2}$$

where N_i are shape functions in the form

$$\begin{aligned} N_1(r,s) &= \tfrac{1}{4}(1+r)(1+s), \qquad & N_2(r,s) &= \tfrac{1}{4}(1-r)(1+s), \\ N_3(r,s) &= \tfrac{1}{4}(1-r)(1-s), \qquad & N_4(r,s) &= \tfrac{1}{4}(1+r)(1-s) \end{aligned} \tag{A3}$$

and $u_i, v_i, \ldots, \beta_{yi}$ are values of the i-th element displacement vector $\mathbf{u}_{el}$ The Cauchy's strains may be written as follows

- membrane strains

$$\boldsymbol{\varepsilon}_m = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^T = \mathbf{B}_m \cdot \mathbf{u}_{el} \tag{A4}$$

- bending strains

$$\boldsymbol{\varepsilon}_b = z \cdot \left[\frac{\partial \beta_x}{\partial x}, -\frac{\partial \beta_y}{\partial y}, \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x}\right]^T = \mathbf{B}_b \cdot \mathbf{u}_{el} \tag{A5}$$

- transverse shear strains

$$\boldsymbol{\varepsilon}_s = \left[\left(\frac{\partial w}{\partial y} - \beta_y\right), \left(\frac{\partial w}{\partial x} + \beta_x\right)\right]^T = \mathbf{B}_s \cdot \mathbf{u}_{el} \tag{A6}$$

where

$$\mathbf{B}_m = \begin{bmatrix} N_{1,X} & 0 & 0 & 0 & 0 & & N_{4X} & 0 & 0 & 0 & 0 \\ 0 & N_{1,Y} & 0 & 0 & 0 & \ldots & 0 & N_{4,Y} & 0 & 0 & 0 \\ N_{1,y} & N_{1,X} & 0 & 0 & 0 & & N_{4,y} & N_{4,X} & 0 & 0 & 0 \end{bmatrix} \tag{A7}$$

$$\mathbf{B}_b = \begin{bmatrix} 0 & 0 & 0 & N_{1,X} & 0 & & 0 & 0 & 0 & N_{4,X} & 0 \\ 0 & 0 & 0 & 0 & -N_{1,Y} & \ldots & 0 & 0 & 0 & 0 & -N_{4,Y} \\ 0 & 0 & 0 & N_{1,Y} & -N_{1,X} & & 0 & 0 & 0 & N_{4,Y} & -N_{4,X} \end{bmatrix} \tag{A8}$$

$$\mathbf{B}_s = \begin{bmatrix} 0 & 0 & N_{1,Y} & 0 & -N_1 & \ldots & 0 & 0 & N_{4,Y} & 0 & -N_4 \\ 0 & 0 & N_{1,X} & N_1 & 0 & & 0 & 0 & N_{4,X} & N_4 & 0 \end{bmatrix}. \tag{A9}$$

The shape functions differentiation with the respect to x or y is

$$\begin{bmatrix} N_{1,X} & 0 & N_{2,X} & 0 & N_{3,X} & 0 & N_{4,X} & 0 \\ N_{1,Y} & 0 & N_{2,Y} & 0 & N_{3,Y} & 0 & N_{4,Y} & 0 \end{bmatrix} = \mathbf{J}^{-1} \cdot \begin{bmatrix} N_{1,r} & 0 & N_{2,r} & 0 & N_{3,r} & 0 & N_{4,r} & 0 \\ N_{1,s} & 0 & N_{2,s} & 0 & N_{3,s} & 0 & N_{4,s} & 0 \end{bmatrix} \tag{A10}$$

and $\mathbf{J}$ is well-known Jacobian matrix which may be written

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \end{bmatrix} \tag{A11}$$

and the shape functions differentiation with the respect to s or r are

$$\begin{aligned} N_{1,r} &= \tfrac{1}{4}\cdot(1+s),\ N_{2,r} = -\tfrac{1}{4}\cdot(1+s),\ N_{3,r} = -\tfrac{1}{4}\cdot(1-s),\ N_{1,r} = \tfrac{1}{4}\cdot(1-s), \\ N_{1,s} &= \tfrac{1}{4}\cdot(1+r),\ N_{2,s} = \tfrac{1}{4}\cdot(1-r),\quad N_{3,s} = -\tfrac{1}{4}\cdot(1-r),\ N_{1,s} = -\tfrac{1}{4}\cdot(1+r). \end{aligned} \tag{A12}$$

As a result, the shell element stiffness matrix can be expressed

$$\mathbf{K}_i = \int_{-1}^{1}\int_{-1}^{1} [(\mathbf{B}_m^T\cdot\mathbf{E}_m\cdot\mathbf{B}_m) + (\mathbf{B}_b^T\cdot\mathbf{E}_b\cdot\mathbf{B}_b) + (\mathbf{B}_s^T\cdot\mathbf{E}_s\cdot\mathbf{B}_s)]\cdot\det(\mathbf{J})\cdot \mathrm{d}r\cdot\mathrm{d}s, \tag{A13}$$

where the material property matrices are given as

$$\begin{aligned} \mathbf{E_m} &= \tfrac{E\cdot t}{1-\mu^2}\cdot\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = t\cdot\mathbf{D}\,, \\ \mathbf{E_b} &= \tfrac{E\cdot t^3}{12\cdot(1-\mu^2)}\cdot\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = \tfrac{t^3}{12}\cdot\mathbf{D}\,, \\ \mathbf{E_s} &= \tfrac{\alpha\cdot E\cdot t}{2\cdot(1+\mu)}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = t\cdot\mathbf{D}_s\,, \end{aligned} \tag{A14}$$

where E and μ represent the elastic modulus and the Poisson's ratio, t is the element thickness, α is a shear correction factor ($\alpha = 5/6$). Calculation of the $\mathbf{K}_i$ can be realized numerically instead of analytically, i.e.

$$\begin{aligned} \mathbf{K}_i \;=\; & \sum_{p=1}^{m}\sum_{q=1}^{m}\alpha_p\cdot\alpha_q\cdot\mathbf{B}_m^T(r_p,s_q)\cdot\mathbf{E}_m\cdot\mathbf{B}_m(r_p,s_q)\cdot\det(\mathbf{J}(r_p,s_q)) + \\ + & \sum_{p=1}^{m}\sum_{q=1}^{m}\alpha_p\cdot\alpha_q\cdot\mathbf{B}_b^T(r_p,s_q)\cdot\mathbf{E}_b\cdot\mathbf{B}_b(r_p,s_q)\cdot\det(\mathbf{J}(r_p,s_q)) + \\ + & \sum_{p=1}^{m}\sum_{q=1}^{m}\alpha_p\cdot\alpha_q\cdot\mathbf{B}_s^T(r_p,s_q)\cdot\mathbf{E}_s\cdot\mathbf{B}_s(r_p,s_q)\cdot\det(\mathbf{J}(r_p,s_q)) \end{aligned} \tag{A15}$$

where m denotes a degree of Gauss integration (usualy $m = 2$), r_p, s_q, are coordinates of integrations points (for $m = 2$, $r_p = s_q = 0.577\,350\,269$) and α_p, α_q are weight coefficients (for $m = 2$, $\alpha_p = \alpha_q = 1.0$). The shear locking efect usually leads to the decrease of the integration degree for the shear part of $\mathbf{K}_i$ [1, 2, 5].